

# Extensions of Lie-Rinehart algebras and cotangent bundle reduction

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## Abstract

Let  $Q$  be a smooth manifold acted upon smoothly by a Lie group  $G$ , and let  $N$  be the space of  $G$ -orbits. The  $G$ -action lifts to an action on the total space  $T^*Q$  of the cotangent bundle of  $Q$  and hence on the ordinary symplectic Poisson algebra of smooth functions on  $T^*Q$ , and the Poisson algebra of  $G$ -invariant functions on  $T^*Q$  yields a Poisson structure on the space  $(T^*Q)/G$  of  $G$ -orbits. We develop a description of this Poisson structure in terms of the orbit space  $N$  and suitable additional data. When the  $G$ -action on  $Q$  is principal, the problem admits a simple solution in terms of extensions of Lie-Rinehart algebras. In the general case, extensions of Lie-Rinehart algebras do not suffice, and we show how the requisite supplementary information can be recovered from invariant theory.

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## 1 Introduction

Let  $Q$  be a smooth manifold acted upon smoothly by a Lie group  $G$ , and let  $N$  be the space of  $G$ -orbits. The  $G$ -action lifts to an action on the total space  $T^*Q$  of the cotangent bundle of  $Q$  and hence on the ordinary symplectic Poisson algebra of smooth functions on  $T^*Q$ . Collective symplectic reduction then leads to the Poisson algebra of  $G$ -invariant

functions on  $T^*Q$  and the question arises to determine this Poisson algebra in terms of the orbit space  $N$  and the requisite additional data.

The simplest case one could perhaps think of, apart from trivial ones, is that where  $Q$  is the group  $G$  itself, with  $G$ -action via left translation. In this case, the Poisson algebra of  $G$ -invariant functions on  $T^*Q$  simply comes down to the familiar Lie-Poisson algebra of smooth functions on the dual  $\mathfrak{g}^*$  of the Lie algebra  $\mathfrak{g}$  of  $G$ . A non-degenerate  $G$ -invariant quadratic form on  $\mathfrak{g}$  then defines a  $G$ -invariant Riemannian metric or, equivalently, kinetic energy on  $Q$ , and the equations of motion on  $Q$  or, equivalently, the equations defining the geodesics, are determined by the Euler-Poincaré equations on  $\mathfrak{g}$ . For  $G = \mathrm{SO}(3, \mathbb{R})$ , viewed as the configuration space of a rigid body, the inertia tensor of the body is precisely such a quadratic form, and the Euler-Poincaré equations are simply the familiar Euler equations for the motion of a rigid body [1], [3], [19].

For more general  $Q$  and  $G$ , in recent years, various attempts have been made to arrive at explicit descriptions of the Poisson algebra of  $G$ -invariant functions on  $T^*Q$ , in the case where the action is free and proper [23], [26], as well as in a certain singular situation [6]. In the present paper we will show that the constructions in the quoted references admit a simple explanation in terms of what we refer to as the *tautological Poisson structure of a Lie-Rinehart algebra* and, furthermore, in terms of extensions of Lie-Rinehart algebras. We have explored this tautological Poisson structure already in our paper [7] (without having introduced that terminology); suffice it to mention at the present stage that, for the special case where the Lie-Rinehart algebra  $(A, L)$  under consideration is the pair  $(C^\infty(M), \mathrm{Vect}(M))$  which consists of the smooth functions  $C^\infty(M)$  and the smooth vector fields  $\mathrm{Vect}(M)$  on a smooth manifold  $M$ , the tautological Poisson algebra comes down to the algebra of smooth functions on the total space  $T^*M$  of the cotangent bundle of  $M$  that are polynomial on the fibers, endowed with the ordinary cotangent bundle Poisson structure.

When the action of  $G$  on  $Q$  is principal, the Poisson algebra of  $G$ -invariant functions on  $T^*Q$  can be completely characterized in terms of the tautological Poisson structure of a suitable Lie-Rinehart algebra and therefore in terms of extensions of Lie-Rinehart algebras; see Corollary 3.4 below. This kind of description generalizes to Lie algebroids with constant rank structure map, cf. Theorem 3.6 below, in particular, to transitive Lie algebroids, as explained in Corollary 3.7 below.

In the non-principal case, this kind of description and hence the descriptions given in the quoted references can only partially recover the Poisson algebra of  $G$ -invariant functions on  $T^*Q$ , and we show how the requisite supplementary information can be obtained from invariant theory, in the same spirit as we exploited invariant theory in [15]. The additional terms include, in particular, kinetic energy terms.

We plan to explore, at another occasion, the significance of our approach for the equations of motion.

## 2 The tautological Poisson algebra of a Lie-Rinehart algebra

The total space  $T^*M$  of the cotangent bundle  $\tau_M^*: T^*M \rightarrow M$  of a smooth finite-dimensional manifold  $M$  is well known to inherit a symplectic Poisson structure  $\{\cdot, \cdot\}$ . The smooth vector fields on  $M$  yield functions on  $T^*M$  that are linear on the fibers and, given two such vector fields  $\alpha$  and  $\beta$  on  $M$ , viewed as functions on  $T^*M$ , their Poisson bracket  $\{\alpha, \beta\}$  is simply the Lie bracket  $[\alpha, \beta]$  of vector fields. Since the vector fields on  $M$  together with the functions on  $T^*M$  that are constant on the leaves of the vertical (i. e. cotangent bundle) foliation, that is, functions that come from functions on  $M$  through the projection to  $M$ , yield enough coordinate functions on  $T^*M$ , the Lie bracket  $[\cdot, \cdot]$  on the Lie algebra  $\text{Vect}(M)$  of smooth vector fields on  $M$  together with the operation  $(\alpha, f) \mapsto \alpha(f)$  of evaluation of a vector field  $\alpha$  in a smooth function  $f$  on  $M$  yields a complete description of the Poisson structure on  $T^*M$ . This observation is of course a version of the standard fact that, relative to the obvious filtration of the algebra  $\text{Diff}(M)$  of globally defined differential operators on  $M$ , the associated graded algebra is the algebra of smooth functions on  $T^*M$  that are polynomial on the fibers.

The pair  $(A, L) = (C^\infty(M), \text{Vect}(M))$  is among the standard examples of a *Lie-Rinehart algebra* [7], [8], [9], [24] and, abstracting from the cotangent bundle situation, we are led to the *tautological Poisson algebra* associated with a Lie-Rinehart algebra:

Let  $R$  be a commutative ring, fixed throughout; the unadorned tensor product symbol  $\otimes$  will always refer to the tensor product over  $R$ . Further, let  $A$  be a commutative  $R$ -algebra. An  $(R, A)$ -Lie algebra [24] is a Lie algebra  $L$  over  $R$  which acts on (the left of)  $A$  by derivations (the action being written as  $(\alpha \otimes a) \mapsto \alpha a$ ), and is also an  $A$ -module (the structure map being written as  $(a \otimes \alpha) \mapsto a\alpha$ ), in such a way that suitable compatibility conditions are satisfied which generalize standard properties of the Lie algebra of vector fields on a smooth manifold viewed as a module over its ring of functions; these conditions read

$$(a\alpha)(b) = a(\alpha(b)), \quad \alpha \in L, a, b \in A, \quad (2.1)$$

$$[\alpha, a\beta] = a[\alpha, \beta] + \alpha(a)\beta, \quad \alpha, \beta \in L, a \in A. \quad (2.2)$$

When the emphasis is on the pair  $(A, L)$ , with the mutual structure of interaction, we refer to a *Lie-Rinehart algebra*. Given two Lie-Rinehart algebras  $(A, L)$  and  $(A', L')$ , a *morphism*  $(\phi, \psi): (A, L) \rightarrow (A', L')$  of *Lie-Rinehart algebras* is the obvious thing, that is,  $\phi$  and  $\psi$  are morphisms in the appropriate categories that are compatible with the additional structure. With this notion of morphism, Lie-Rinehart algebras constitute a category. Apart from the example of smooth functions and smooth vector fields on a smooth manifold, a related (but more general) example is the pair consisting of a commutative algebra  $A$  and the  $R$ -module  $\text{Der}(A)$  of derivations of  $A$  with the obvious  $A$ -module structure; here the commutativity of  $A$  is crucial.

Let  $(A, L)$  be a Lie-Rinehart algebra; thus  $A$  is a commutative algebra and  $L$  an  $(R, A)$ -Lie algebra. Let  $\mathcal{S}_A[L]$  be the symmetric  $A$ -algebra on  $L$ . The  $L$ -action on  $A$  and the bracket operation on  $L$  induce an obvious Poisson structure

$$\{\cdot, \cdot\}: \mathcal{S}_A[L] \otimes \mathcal{S}_A[L] \rightarrow \mathcal{S}_A[L] \quad (2.3)$$

on  $\mathcal{S}_A[L]$ ; explicitly, this structure is determined by

$$\{\alpha, \beta\} = [\alpha, \beta], \quad \alpha, \beta \in L, \quad (2.4)$$

$$\{\alpha, a\} = \alpha(a) \in A, \quad a \in A, \alpha \in L, \quad (2.5)$$

$$\{u, vw\} = \{u, v\}w + v\{u, w\}, \quad u, v, w \in \mathcal{S}_A[L]. \quad (2.6)$$

We refer to the Poisson algebra  $(\mathcal{S}_A[L], \{\cdot, \cdot\})$  as the *tautological Poisson algebra associated with the Lie-Rinehart algebra  $(A, L)$* . The commutative  $R$ -algebra  $A$  being fixed, the tautological Poisson algebra is plainly functorial in  $(R, A)$ -Lie algebras: A morphism  $\phi: L_1 \rightarrow L_2$  of  $(R, A)$ -Lie algebras induces a morphism

$$(\mathcal{S}_A[L_1], \{\cdot, \cdot\}) \longrightarrow (\mathcal{S}_A[L_2], \{\cdot, \cdot\}) \quad (2.7)$$

of Poisson algebras.

We will now justify the terminology “tautological Poisson algebra”. To this end, we recall briefly how, for an arbitrary Poisson algebra, an appropriate Lie-Rinehart algebra serves as a replacement for the tangent bundle of a smooth symplectic manifold.

Let  $(\mathcal{A}, \{\cdot, \cdot\})$  be a Poisson algebra, and let  $D_{\mathcal{A}}$  be the  $\mathcal{A}$ -module of formal differentials of  $\mathcal{A}$  the elements of which we write as  $du$ , for  $u \in \mathcal{A}$ . Here we use the notation  $\mathcal{A}$  for a general Poisson algebra to avoid conflict with the notation  $A$  for an algebra which, below, will not be a Poisson algebra, and to avoid conflict with the notation  $\mathcal{S}_A[L]$  as well. For  $u, v \in \mathcal{A}$ , the assignment to  $(du, dv)$  of  $\pi(du, dv) = \{u, v\}$  yields an  $\mathcal{A}$ -valued  $\mathcal{A}$ -bilinear skew-symmetric 2-form

$$\pi = \pi_{\{\cdot, \cdot\}}: D_{\mathcal{A}} \otimes D_{\mathcal{A}} \longrightarrow \mathcal{A} \quad (2.8)$$

on  $D_{\mathcal{A}}$ , the *Poisson 2-form* for  $(\mathcal{A}, \{\cdot, \cdot\})$ . Its adjoint

$$\pi^{\sharp}: D_{\mathcal{A}} \longrightarrow \text{Der}(\mathcal{A}) = \text{Hom}_{\mathcal{A}}(D_{\mathcal{A}}, \mathcal{A}) \quad (2.9)$$

is a morphism of  $\mathcal{A}$ -modules, and the formula

$$[adu, bdv] = a\{u, b\}dv + b\{a, v\}du + abd\{u, v\} \quad (2.10)$$

yields a Lie bracket  $[\cdot, \cdot]$  on  $D_{\mathcal{A}}$ , viewed as an  $R$ -module. More details can be found in [7]. For the record we recall the following, established in [7] (3.8).

**Proposition 2.1.** *The  $\mathcal{A}$ -module structure on  $D_{\mathcal{A}}$ , the bracket  $[\cdot, \cdot]$ , and the morphism  $\pi^{\sharp}$  of  $\mathcal{A}$ -modules turn the pair  $(\mathcal{A}, D_{\mathcal{A}})$  into a Lie-Rinehart algebra in such a way that  $\pi^{\sharp}$  is a morphism of Lie-Rinehart algebras.*

We will use the notation  $D_{\{\cdot, \cdot\}}$  for the  $(R, \mathcal{A})$ -Lie algebra  $(D_{\mathcal{A}}, [\cdot, \cdot], \pi^{\sharp})$ . The 2-form  $\pi_{\{\cdot, \cdot\}}$ , which is defined for *every* Poisson algebra, is plainly a 2-cocycle in the Rinehart algebra  $(\text{Alt}_{\mathcal{A}}(D_{\{\cdot, \cdot\}}, \mathcal{A}), d)$ . In [7], we defined the *Poisson cohomology*  $H_{\text{Poisson}}^*(\mathcal{A}, \mathcal{A})$  of the Poisson algebra  $(\mathcal{A}, \{\cdot, \cdot\})$  to be the cohomology of this Rinehart algebra, that is,

$$H_{\text{Poisson}}^*(\mathcal{A}, \mathcal{A}) = H^*(\text{Alt}_{\mathcal{A}}(D_{\{\cdot, \cdot\}}, \mathcal{A}), d). \quad (2.11)$$

When  $\mathcal{A}$  is the algebra of smooth functions on a smooth symplectic manifold  $M$ , endowed with the symplectic Poisson structure, a smooth version  $D_{\mathcal{A}}^{\text{smooth}}$  of  $D_{\mathcal{A}}$  comes down to the space  $\Gamma(\tau_M^*)$  of smooth sections of the cotangent bundle  $\tau_M^*$  of  $M$ ; the inverse of the symplectic structure then induces a vector bundle isomorphism from  $\tau_M^*$  onto  $\tau_M$  and hence an isomorphism from the smooth version  $D_{\{\cdot, \cdot\}}^{\text{smooth}} \cong \Omega^1(M)$  (the  $C^\infty$ -module of ordinary 1-forms on  $M$ ) of  $D_{\{\cdot, \cdot\}}$  onto the ordinary  $(\mathbb{R}, \mathcal{A})$ -Lie algebra of smooth vector fields on  $M$  which identifies the 2-form  $\pi$  with the symplectic structure. Furthermore, under the canonical isomorphism between  $\text{Alt}_{\mathcal{A}}(D_{\mathcal{A}}^{\text{smooth}}, \mathcal{A})$  and the exterior  $\mathcal{A}$ -algebra  $\Lambda_{\mathcal{A}}(\text{Vect}(M))$  on  $\text{Vect}(M)$ , the 2-form  $\pi$  corresponds to the familiar Poisson 2-tensor. The 2-form  $\pi$  is defined for an arbitrary Poisson algebra, though, whether or not it arises from a smooth symplectic manifold. See [7] for details.

The Poisson 2-form  $\pi: D_{\mathcal{S}} \otimes_{\mathcal{S}} D_{\mathcal{S}} \rightarrow \mathcal{S}$  given by (2.8), for the algebra  $\mathcal{S} = \mathcal{S}_A[L]$ , endowed with the Poisson structure (2.3), is a Poisson coboundary, i. e.  $\{\cdot, \cdot\}$  admits a Poisson potential [7]. Indeed, as an algebra,  $\mathcal{S}$  is generated by the elements of  $A$  and those of  $L$ . Hence as an  $\mathcal{S}$ -module,  $D_{\mathcal{S}}$  is generated by the formal differentials  $da$ ,  $a \in A$ , and  $d\alpha$ ,  $\alpha \in L$ . Let  $D_{\mathcal{S}|A}$  denote the  $\mathcal{S}$ -module of formal differentials of  $\mathcal{S}$  over  $A$ ; as an  $\mathcal{S}$ -module,  $D_{\mathcal{S}|A}$  amounts to an induced module of the kind  $\mathcal{S} \otimes_A L$ , generated by the formal differentials  $d\alpha$  where  $\alpha$  ranges over  $L$ . The sequence of inclusions  $R \subseteq A \subseteq \mathcal{S}$  determines the exact sequence

$$\mathcal{S} \otimes_A D_A \longrightarrow D_{\mathcal{S}} \longrightarrow D_{\mathcal{S}|A} \longrightarrow 0 \quad (2.12)$$

of  $\mathcal{S}$ -modules, and the assignment to the formal differential  $d\alpha$  over  $A$  where  $\alpha$  ranges over  $L$  of  $d\alpha$  viewed as a formal differential over  $A$  yields an  $\mathcal{S}$ -module splitting of (2.12) where the notation  $d\alpha$  is slightly abused. A straightforward calculation, cf. p. 92 of [7], shows that the 1-form  $\vartheta: D_{\mathcal{S}} \rightarrow \mathcal{S}$  given through the projection to  $D_{\mathcal{S}|A}$  by

$$\vartheta(d\alpha) = \alpha, \quad \alpha \in L, \quad (2.13)$$

is a Poisson potential for  $\{\cdot, \cdot\}$ , that is

$$\pi = d\vartheta \in \text{Alt}_{\mathcal{S}}^2(D_{\{\cdot, \cdot\}}, \mathcal{S}). \quad (2.14)$$

For later reference, we spell out  $\pi$  explicitly: Consider the obvious  $\mathcal{S}$ -module surjection

$$\mathcal{S} \otimes_A D_A \oplus D_{\mathcal{S}|A} \longrightarrow D_{\mathcal{S}}. \quad (2.15)$$

Under suitable circumstances, this surjection is actually an isomorphism, e. g. when  $L$  is projective as an  $A$ -module. In the general case, we will not distinguish in notation between  $\pi$  and the corresponding 2-form on the left-hand side of (2.15). With these preparations out of the way, the 2-form  $\pi$  is simply given by

$$\pi(d\alpha, d\beta) = [\alpha, \beta], \quad \alpha, \beta \in L, \quad (2.16)$$

$$\pi(d\alpha, db) = \alpha(b), \quad \alpha \in L, \quad b \in A. \quad (2.17)$$

In particular, the following observation is immediate; yet we spell it out, for later reference:

**Proposition 2.2.** *The Poisson structure (2.3) on  $\mathcal{S}_A[L]$  recovers the Lie-Rinehart structure on  $(A, L)$ , that is, this Lie-Rinehart structure can be reconstructed from the Poisson structure.*

When  $\mathcal{A}$  is the standard symplectic Poisson algebra of smooth functions on the total space  $M = T^*N$  of the cotangent bundle on a smooth manifold  $N$ , the smooth version of the present 1-form  $\vartheta$  amounts to the *tautological* 1-form on  $M = T^*N$ , and the resulting symplectic structure is occasionally referred to as the *tautological symplectic structure* on  $M = T^*N$ . Actually, in this case, with the notation  $(A, L) = (C^\infty(N), \text{Vect}(N))$ , in the appropriate Fréchet topology,  $\mathcal{S}_A[L]$  is dense in the algebra  $C^\infty(T^*N)$  of smooth functions on  $M = T^*N$ . Under the canonical isomorphism between  $\text{Alt}_{\mathcal{A}}(\mathcal{D}_{\mathcal{A}}^{\text{smooth}}, \mathcal{A}) = \text{Alt}_{\mathcal{A}}(\Omega^1(M), \mathcal{A})$  and the exterior  $\mathcal{A}$ -algebra  $\Lambda_{\mathcal{A}}(\text{Vect}(M))$  on  $\text{Vect}(M)$  spelled out earlier for a general smooth symplectic manifold  $M$ , the formulas (2.16) and (2.17) then yield the Poisson 2-tensor for the cotangent bundle Poisson structure on  $T^*N$ . The Poisson structure (2.3) on the algebra  $\mathcal{S}_A[L]$  relative to a general Lie-Rinehart algebra  $(A, L)$  is formally of the same kind. This is the reason for the terminology *tautological Poisson structure*. For an ordinary Lie algebra  $\mathfrak{g}$  over the ground ring  $R$ , viewed as an  $(R, R)$ -Lie algebra with trivial  $\mathfrak{g}$ -action on  $R$ , the tautological Poisson structure plainly comes down to the algebraic version of the ordinary Lie-Poisson structure on the dual  $\mathfrak{g}^*$ .

### 3 Extensions of Lie-Rinehart algebras and tautological Poisson algebra

The algebraic analogue of an “Atiyah sequence” or of a “transitive Lie algebroid” (see below for details) is an extension of Lie-Rinehart algebras [11].

Let  $L', L, L''$  be  $(R, A)$ -Lie algebras. An *extension* of  $(R, A)$ -Lie algebras is a short exact sequence

$$\mathbf{e}: 0 \longrightarrow L' \longrightarrow L \xrightarrow{p} L'' \longrightarrow 0 \quad (3.1)$$

in the category of  $(R, A)$ -Lie algebras; notice in particular that the Lie algebra  $L'$  necessarily acts trivially on  $A$ . If also  $\bar{\mathbf{e}}: 0 \rightarrow L' \rightarrow \bar{L} \rightarrow L'' \rightarrow 0$  is an extension of  $(R, A)$ -Lie algebras, as usual,  $\mathbf{e}$  and  $\bar{\mathbf{e}}$  are said to be *congruent* whenever there is a morphism  $(\text{Id}, \cdot, \text{Id}): \mathbf{e} \longrightarrow \bar{\mathbf{e}}$  of extensions of  $(R, A)$ -Lie algebras.

**Example 3.1.** *Let the ground ring  $R$  be the field  $\mathbb{R}$  of real numbers, let  $N$  be a smooth finite dimensional manifold, let  $A$  be the algebra of smooth functions on  $N$ , let  $G$  be a Lie group, and let  $\pi: Q \rightarrow N$  be a principal  $G$ -bundle. The vertical subbundle  $\psi: V \rightarrow Q$  of the tangent bundle  $\tau_Q$  of  $Q$  is well known to be trivial (beware, not equivariantly trivial), having as fibre the Lie algebra  $\mathfrak{g}$  of  $G$ , that is,  $V \cong Q \times \mathfrak{g}$ . Dividing out the  $G$ -actions, we obtain an extension*

$$0 \longrightarrow \text{ad}(\pi) \longrightarrow \tau_Q/G \longrightarrow \tau_N \longrightarrow 0 \quad (3.2)$$

*of vector bundles over  $N$ , where  $\tau_N$  is the tangent bundle of  $N$ . This sequence has been introduced by Atiyah [4] and is now usually referred to as the Atiyah sequence of the principal bundle  $\pi$ ; here  $\text{ad}(\pi)$  is the bundle associated to the principal bundle by the adjoint representation of  $G$  on its Lie algebra  $\mathfrak{g}$ . The spaces  $\mathfrak{g}(\pi) = \Gamma \text{ad}(\pi)$  and  $E(\pi) = \Gamma(\tau_Q/G)$*

of smooth sections inherit obvious Lie algebra structures, in fact  $(\mathbb{R}, A)$ -Lie algebra structures, and

$$0 \longrightarrow \mathfrak{g}(\pi) \longrightarrow E(\pi) \longrightarrow \text{Vect}(N) \longrightarrow 0 \quad (3.3)$$

is an extension of  $(\mathbb{R}, A)$ -Lie algebras; here  $\text{Vect}(N) = \Gamma(\tau_N)$ , the Lie algebra of vector fields on  $N$ , and  $\mathfrak{g}(\pi)$  is in an obvious way the Lie algebra of the group of gauge transformations of  $\pi$ . The resulting Lie algebroid

$$\tau_Q/G: (TQ)/G \longrightarrow N \quad (3.4)$$

over  $N$  is plainly transitive.

We note that transitive Lie algebroids which do *not* arise from a principal bundle abound, though, cf. e. g. [2].

We now return to the situation where the ground ring  $R$  is general. The classical notions of principal connection and curvature generalize to extensions of Lie-Rinehart algebras in an obvious manner: Let  $\mathbf{e}$  be an extension of  $(R, A)$ -Lie algebras of the kind (3.1), and suppose that it splits in the category of  $A$ -modules; this will e. g. be so whenever  $L''$  is projective as an  $A$ -module. Then  $\mathbf{e}$  can be represented by a Lie algebra 2-cocycle: Let  $\omega: L'' \rightarrow L'$  be a section of  $A$ -modules for the projection  $p: L \rightarrow L''$ . We refer to  $\omega$  as an  $\mathbf{e}$ -connection. Given an  $\mathbf{e}$ -connection, define the corresponding ( $\mathbf{e}$ -)curvature  $\Omega: L'' \otimes_A L'' \rightarrow L'$  as the morphism  $\Omega$  of  $A$ -modules satisfying the identity

$$[\omega(\alpha), \omega(\beta)] = \omega[\alpha, \beta] + \Omega(\alpha, \beta) \quad (3.5)$$

for every  $\alpha, \beta \in L''$ . The standard reasoning reveals that  $\Omega$  is indeed well defined as an alternating  $A$ -bilinear 2-form on  $L''$  with values in  $L'$ ; under the circumstances of the Example 3.1, this amounts to  $\Omega$  being a tensor. For a general extension (3.1) of  $(R, A)$ -Lie algebras, a choice of connection  $\omega$  induces an  $A$ -module decomposition

$$L = L' \oplus \omega(L'') \cong L' \oplus L''$$

and, in terms of this decomposition, the Lie bracket  $[\cdot, \cdot]$  on  $L$  takes the form

$$[(\alpha', \alpha''), (\beta', \beta'')] = ([\alpha', \beta'] + \Omega(\alpha'', \beta''), [\alpha'', \beta'']). \quad (3.6)$$

Here and below the brackets  $[\alpha', \beta']$  and  $[\alpha'', \beta'']$  refer to the brackets in  $L'$  and  $L''$ , respectively, with a slight abuse of the bracket notation  $[\cdot, \cdot]$ .

The notions of  $\mathbf{e}$ -connection and  $\mathbf{e}$ -curvature manifestly generalize the concepts of principal connection and principal curvature; indeed, under the circumstances of the Example 3.1, they come down to their descriptions in the language of Atiyah sequences.

**Theorem 3.2.** *Given an extension  $\mathbf{e}: 0 \longrightarrow L' \longrightarrow L \longrightarrow L'' \longrightarrow 0$  of  $(R, A)$ -Lie algebras of the kind (3.1) together with an  $\mathbf{e}$ -connection  $\omega: L'' \rightarrow L'$  having  $\mathbf{e}$ -curvature  $\Omega: L'' \otimes_A L'' \rightarrow L'$ , the tautological Poisson structure*

$$\{\cdot, \cdot\}: \mathcal{S}_A[L] \otimes \mathcal{S}_A[L] \longrightarrow \mathcal{S}_A[L]$$

on  $\mathcal{S}_A[L]$  is determined by the identities

$$\{\alpha', \beta'\} = [\alpha', \beta'], \quad \alpha', \beta' \in L', \quad (3.7)$$

$$\{\alpha'', \beta''\} = [\alpha'', \beta''] + \Omega(\alpha'', \beta''), \quad \alpha'', \beta'' \in L'', \quad (3.8)$$

$$\{\alpha'', a\} = \alpha''(a) \in A, \quad a \in A, \alpha'' \in L'', \quad (3.9)$$

$$\{\alpha', a\} = 0 \in A, \quad a \in A, \alpha' \in L', \quad (3.10)$$

$$\{u, vw\} = \{u, v\}w + v\{u, w\}, \quad u, v, w \in \mathcal{S}_A[L]. \quad (3.11)$$

*Proof.* This is an immediate consequence of the identities (2.4)–(2.6), combined with the identity (3.6).  $\square$

**Corollary 3.3.** *Under the circumstances of Theorem 3.2, the extension  $\mathbf{e}$  of  $(R, A)$ -Lie algebras can be reconstructed from the Poisson structure on  $\mathcal{S}_A[L]$ .*

*Proof.* This is an immediate consequence of Theorem 3.2 combined with Proposition 2.2.  $\square$

We will now specialize again to the case where the ground ring  $R$  is the field  $\mathbb{R}$  of real numbers. The following is immediate, but we spell it out for the sake of reference.

**Corollary 3.4.** *Under the circumstances of the Example 3.1, the description of the tautological Poisson bracket (3.2) in Theorem 3.2 completely recovers the Poisson structure on  $(T^*Q)/G$ .*

Indeed, this description of the Poisson structure on  $(T^*Q)/G$  amounts to that given in Theorem IV.1 of [26]: Once a choice of connection on the principal bundle  $Q \rightarrow N$  has been made, the description of the Poisson structure given there involves three terms, one coming from the ordinary cotangent bundle Poisson structure on  $T^*N$ , one coming from the Lie-Poisson structure on  $\mathfrak{g}^*$ , and a third term determined by the curvature of the chosen connection on the principal bundle  $Q \rightarrow N$ . The first term in the description given in [26] corresponds to the first term on the right-hand side of (3.8) and the identity (3.9) above, the second term in the description given in [26] corresponds to the second term on the right-hand side of (3.8)—indeed, this is the curvature term—, and the third term in the description given in [26] corresponds to the identity (3.7). Our approach provides a simple explanation for this Poisson structure: The two terms in the Poisson tensor beyond that coming from the cotangent bundle Poisson structure on the base  $N$  simply reconstruct the Atiyah sequence (3.2) of the principal bundle or, equivalently, the corresponding extension of  $(\mathbb{R}, C^\infty(N))$ -Lie algebras, in terms of the connection.

Since the extension (3.3) of  $(\mathbb{R}, C^\infty(N))$ -Lie algebras arises from a principal bundle, this extension has peculiar features, which we will now spell them out, to distinguish this case from the singular situation explored later in the paper, where these features will not survive:

Introduce the notation  $(A_Q, L_Q) = (C^\infty(Q), \text{Vect}(Q))$ . The induced projections  $TQ \rightarrow TQ/G$  and  $T^*Q \rightarrow T^*Q/G$  are principal  $G$ -bundles, and the  $(\mathbb{R}, A)$ -Lie algebra  $L$  amounts to the space of smooth sections of the Lie algebroid (3.4) or, equivalently, to the space  $L_Q^G$  of smooth  $G$ -equivariant vector fields on  $Q$ . The canonical injection of



$C^\infty((T^*Q)/G)$  into  $(C^\infty(T^*Q))^G$  is an isomorphism of  $\mathbb{R}$ -algebras whence the canonical injection of  $\mathcal{S}_A[L]$  into  $(\mathcal{S}_{A_Q}[L_Q])^G$  is exhaustive, that is, this injection is an isomorphism

$$\mathcal{S}_A[L] = \mathcal{S}_A[L_Q^G] \longrightarrow (\mathcal{S}_{A_Q}[L_Q])^G \quad (3.12)$$

of  $\mathbb{R}$ -algebras as well. Hence the injection of  $\mathcal{S}_A[L]$  into  $C^\infty(T^*Q)^G$  yields enough coordinate functions on the quotient space  $(T^*Q)/G$  for a complete description of the induced smooth Poisson structure on  $(T^*Q)/G$ . Furthermore, the induced surjection

$$(\mathcal{S}_A[L], \{\cdot, \cdot\}) \longrightarrow (\mathcal{S}_A[L''], \{\cdot, \cdot\}) \quad (3.13)$$

of Poisson algebras, cf. (2.7), is plainly the induced map of Poisson algebras induced by the canonical injection  $T^*N \rightarrow (T^*Q)/G$  of smooth Poisson manifolds,  $T^*N$  being the Marsden-Weinstein reduced space  $\mu^{-1}(0)/G$  at zero momentum relative to the standard momentum mapping  $\mu: T^*Q \rightarrow \mathfrak{g}^*$ .

More generally, given the coadjoint orbit  $\mathcal{O} \subseteq \mathfrak{g}^*$ , under the canonical Poisson injection of the reduced space  $\mu^{-1}(\mathcal{O})/G$  into  $(T^*Q)/G$ , the Poisson algebra  $(\mathcal{S}_A[L], \{\cdot, \cdot\})$  maps onto a Poisson algebra of smooth functions on the reduced space  $\mu^{-1}(\mathcal{O})/G$  containing enough coordinate functions to yield a complete description of the ordinary smooth symplectic Poisson algebra of that reduced space.

**Remark 3.5.** *The algebra  $\mathcal{S}_A[L]$  amounts to the algebra of smooth  $G$ -invariant functions from  $Q$  to  $\mathcal{S}[\mathfrak{g}]$  and hence contains the algebra  $\mathcal{S}[\mathfrak{g}]^G$  of invariant polynomials in an obvious way. Under a suitable additional hypothesis, the familiar fact that the Poisson manifold  $(T^*Q)/G$  is foliated into symplectic leaves can be interpreted as the statement that the Poisson structure on  $(T^*Q)/G$  is a symplectic structure over the field of Casimir elements. Thus suppose that  $G$  is reductive. In view of a classical result of Chevalley's, the algebra  $\mathcal{S}[\mathfrak{g}]^G$  of invariant polynomials is then itself a polynomial algebra, say, with generators  $y_1, \dots, y_m$ , the algebraic quotient  $\mathfrak{g}^*/G$ —beware, this is not the space of ordinary  $G$ -orbits in  $\mathfrak{g}^*$ —is a copy of real affine space  $\mathbb{R}^m$  which, when  $G$  is compact, contains the ordinary space  $\mathfrak{g}^*/G$  of  $G$ -orbits in  $\mathfrak{g}^*$  as a semi-algebraic set via the projection from  $\mathfrak{g}^*$  to  $\mathbb{R}^m$  (the Hilbert map from  $\mathfrak{g}^*$  to  $\mathbb{R}^m$  given by the chosen generators  $y_1, \dots, y_m$ ). Whether or not  $G$  is compact, the smooth functions in the variables  $y_1, \dots, y_m$ , viewed as functions on  $(T^*Q)/G$ , are Casimir functions. Let  $\mathbb{R}(y_1, \dots, y_m)$  be the field of fractions of the affine coordinate ring  $\mathbb{R}[y_1, \dots, y_m]$  of  $\mathfrak{g}^*/G$  and extend the ground field from  $\mathbb{R}$  to  $\mathbb{R}(y_1, \dots, y_m)$ . Over this field, the Poisson structure on  $(T^*Q)/G$  becomes symplectic, i. e. essentially boils down, at any point  $\alpha_q$  of  $(T^*Q)/G$ , to the symplectic structure, at that point, of the symplectic leaf through that point.*

Finally,  $\lambda: \mathcal{L} \rightarrow N$  be a general Lie algebroid, and let  $(A, L) = (C^\infty(N), \Gamma(\lambda))$  be the associated Lie-Rinehart algebra. For example,  $\lambda$  could be a Lie algebroid of the kind reproduced in Example 3.1. For a general Lie algebroid  $\lambda$ ,  $\mathcal{S}_A[L]$  is in an obvious manner the algebra of smooth functions on the total space  $\mathcal{L}^*$  of the dual bundle  $\lambda^*: \mathcal{L}^* \rightarrow N$  that are polynomial on the fibers of  $\lambda^*$ , the algebra  $C^\infty(\mathcal{L}^*)$  of ordinary smooth functions on  $\mathcal{L}^*$  acquires a Poisson structure in an obvious way, and the injection of  $\mathcal{S}_A[L]$  into  $C^\infty(\mathcal{L}^*)$  maps the former algebra onto a Fréchet dense subalgebra of the latter and is

plainly compatible with the Poisson structures. The special case of the tangent bundle has been mentioned already. In the general case, the image  $L''$  of  $L$  under the morphism  $L \rightarrow \text{Vect}(N)$  of  $(\mathbb{R}, A)$ -Lie algebras which is part of the Lie algebroid structure of  $\lambda$  is an  $(\mathbb{R}, A)$ -Lie algebra, and the morphism  $L \rightarrow \text{Vect}(N)$  of  $(\mathbb{R}, A)$ -Lie algebras induces a morphism  $\mathcal{S}_A[L] \rightarrow \mathcal{S}_A[\text{Vect}(N)]$  of Poisson algebras, the latter algebra being the ordinary Poisson algebra of smooth functions on the total space  $T^*N$  of the cotangent bundle of  $N$  that are polynomial on the fibers. Moreover, the projection from  $L$  to  $L''$  fits into an extension

$$\mathbf{e}: 0 \longrightarrow L' \longrightarrow L \longrightarrow L'' \longrightarrow 0 \quad (3.14)$$

of  $(\mathbb{R}, A)$ -Lie algebras of the kind (3.1).

Under a suitable additional hypothesis, the construction in the previous section yields the following description of the Poisson algebra  $\mathcal{S}_A[L]$  in terms of  $A$  and the extension (3.14):

**Theorem 3.6.** *Suppose that the structure map  $\lambda: \mathcal{L} \rightarrow TN$  has constant rank, so that  $L''$  defines a foliation  $\mathcal{F}$  on  $N$ , and denote by  $A^{\mathcal{F}}$  the algebra of smooth functions on  $N$  that are constant on the leaves of  $\mathcal{F}$ . Then the Poisson algebras  $\mathcal{S}_A[L]$  and  $\mathcal{S}_A[L'']$  are Poisson algebras even over  $A^{\mathcal{F}}$  (i. e. the functions in  $A^{\mathcal{F}}$  are Casimir functions), the extension (3.14) splits in the category of  $A$ -modules, that is, admits a connection, and the identities (3.7)–(3.11) yield an explicit description of the Poisson algebra  $\mathcal{S}_A[L]$  of smooth functions on  $\mathcal{L}^*$ .*

*Proof.* Since the structure map  $\lambda: \mathcal{L} \rightarrow TN$  has constant rank,  $\lambda$  maps  $\mathcal{L}$  onto the total space of a subbundle of the tangent bundle of  $N$ . As an  $A$ -module,  $L''$  is the space of sections of this vector bundle whence  $L''$  is necessarily projective as an  $A$ -module. This implies the assertion.  $\square$

Here is a special case.

**Corollary 3.7.** *Suppose that  $\lambda: \mathcal{L} \rightarrow TN$  is a transitive Lie algebroid. Then  $L''$  coincides with the  $(\mathbb{R}, A)$ -Lie algebra  $\text{Vect}(N)$  of ordinary smooth vector fields on  $N$ , the extension (3.14) splits in the category of  $A$ -modules, that is, admits a connection, and the identities (3.7)–(3.11) yield an explicit description of the Poisson algebra  $\mathcal{S}_A[L]$  of smooth functions on  $\mathcal{L}^*$  and hence a complete description of the Poisson algebra  $(C^\infty(\mathcal{L}^*), \{\cdot, \cdot\})$  of ordinary smooth functions on  $\mathcal{L}^*$ .*

The Corollary applies, in particular, to the transitive Lie algebroid (3.4), but that special case has been dealt with before.

## 4 Non-regular quotients of cotangent bundles

Let  $Q$  be a smooth manifold, and let  $G$  be a group acting smoothly on  $Q$ , that is, the  $G$ -action lifts to a smooth action on the total space  $TQ$  of  $Q$  turning the tangent bundle  $\tau_Q: TQ \rightarrow Q$  of  $Q$  into a smooth  $G$ -vector bundle. We will henceforth suppose that the action of  $G$  on  $Q$  is not principal, and we will then refer to the quotient  $(T^*Q)/G$  as being *non-regular*. In the rest of the paper we will show that, in this case, the algebra

written as  $\mathcal{S}_A[L]$  in Theorem 3.2 above does not suffice to recover the functions on the quotient  $(T^*Q)/G$ , that is, even though  $\mathcal{S}_A[L]$  embeds into  $(C^\infty(T^*Q))^G$  in an obvious way, this embedding cannot be onto a Fréchet dense subalgebra. Consequently Theorem 3.2 no longer suffices to recover the Poisson algebra on the quotient  $(T^*Q)/G$ .

Let  $N$  be the orbit space  $Q/G$ , endowed with the quotient topology, let  $\pi: Q \rightarrow N$  denote the canonical projection, and let  $(A_Q, L_Q) = (C^\infty(Q), \text{Vect}(Q))$ . Moreover, let  $A = C^\infty(Q)^G$ , the algebra of smooth  $G$ -invariant functions on  $Q$ , viewed as an algebra of continuous functions on  $N$ , and let  $L = \text{Vect}(Q)^G$ , the Lie algebra of smooth  $G$ -invariant vector fields on  $Q$ . The Lie-Rinehart structure on  $(A_Q, L_Q)$  induces a Lie-Rinehart structure on  $(A, L)$ , in particular, an action  $L \rightarrow \text{Der}(A)$  of  $L$  on  $A$  by derivations. Let  $L''$  denote the image of  $L$  in  $\text{Der}(A)$ . The pair  $(A, L'')$  acquires likewise a Lie-Rinehart structure, and the surjection from  $L$  to  $L''$  fits into an extension

$$e: 0 \longrightarrow L' \longrightarrow L \longrightarrow L'' \longrightarrow 0 \quad (4.1)$$

of  $(\mathbb{R}, A)$ -Lie algebras of the kind (3.1). When  $G$  is a Lie group and when the  $G$ -action on  $Q$  is principal, the extension (4.1) comes down to the Atiyah-sequence of the kind (3.2). In the general case, the tautological Poisson algebras  $\mathcal{S}_A[L]$  and  $\mathcal{S}_A[L'']$  yield Poisson algebras of continuous functions on  $(T^*Q)/G$  and  $T^*N$ , respectively and, by construction,  $\mathcal{S}_A[L]$  embeds into  $(C^\infty(T^*Q))^G$ . Thus, provided the extension (4.1) splits in the category of  $A$ -modules, Theorem 3.2 applies and furnishes an explicit description of the induced Poisson structure on the algebra  $\mathcal{S}_A[L]$  of continuous functions on the quotient space  $(T^*Q)/G$ .

**Theorem 4.1.** *Suppose that the orbit space  $N = Q/G$  is a smooth manifold in such a way that the projection  $\pi: Q \rightarrow N$  is a smooth submersion. Then the algebra  $A$  coincides with the algebra  $C^\infty(N)$  of ordinary smooth functions on  $N$ , the  $(\mathbb{R}, A)$ -Lie algebra  $L''$  amounts to the  $(\mathbb{R}, A)$ -Lie algebra  $\text{Vect}(N)$  of ordinary smooth vector fields on  $N$ , the extension (4.1) splits in the category of  $A$ -modules, that is, admits a connection, and the identities (3.7)–(3.11) yield an explicit description of the induced Poisson algebra  $\mathcal{S}_A[L]$  of continuous functions on the quotient space  $(T^*Q)/G$ .*

A special case of Theorem 4.1 arises when  $G$  is a Lie group and when the action is principal. This case has been dealt with in Theorem 3.6 above. In particular, the induced Poisson algebra  $\mathcal{S}_A[L]$  then yields a complete description of the Poisson algebra  $(C^\infty(T^*Q)^G, \{\cdot, \cdot\})$  on the quotient space  $(T^*Q)/G$ . In Theorem 4.1, we do *not* assert that, for general  $G$  and  $Q$ , a *complete* description of the Poisson algebra  $(C^\infty(T^*Q)^G, \{\cdot, \cdot\})$  on the quotient space  $(T^*Q)/G$  is obtained, though; only the subalgebra  $\mathcal{S}_A[L]$  is recovered, and in general the relationship between  $\mathcal{S}_A[L]$  and  $C^\infty(T^*Q)^G$  is more subtle than that between the functions on the total space of a cotangent bundle that are polynomial on the fibers and all smooth functions on that total space; in particular, the algebra  $\mathcal{S}_A[L]$  does not simply come down to a Fréchet dense subalgebra of  $C^\infty(T^*Q)^G$ , and invariants that cannot be recovered from  $\mathcal{S}_A[L]$  show up. We shall illustrate this fact in Sections 5 and 6 below.

*Proof.* The algebra  $A$  plainly coincides with the algebra  $C^\infty(N)$  of ordinary smooth functions on  $N$  and the regularity assumption implies that the injection from  $L''$  into the  $(\mathbb{R}, A)$ -Lie algebra  $\text{Der}(A) \cong \text{Vect}(N)$  of ordinary smooth vector fields on  $N$  is surjective

as well. Since  $\text{Vect}(N)$  is a projective  $A$ -module, the extension (4.1) splits in the category of  $A$ -modules.  $\square$

Under the circumstances of Theorem 4.1, even though the quotient  $(\mathbb{R}, A)$ -Lie algebra  $L''$  is the  $(\mathbb{R}, A)$ -Lie algebra of smooth vector fields on an ordinary smooth manifold and hence arises from an ordinary Lie algebroid over  $N$ , the  $(\mathbb{R}, A)$ -Lie algebra  $L$  is not necessarily projective as an  $A$ -module and hence does not necessarily arise from an ordinary Lie algebroid over  $N$  unless the  $G$ -action is principal.

For illustration, suppose that the  $G$ -action on  $Q$  involves a single orbit type with compact stabilizer; for example, this situation arises when the action is proper with a single orbit type. Then the orbit space  $N = Q/G$  is a smooth manifold, the projection  $\pi$  is a smooth submersion, cf. e. g. [5] (Theorem 1) where this is established for proper actions preserving a Riemannian metric—we will justify the present more general claim below—, and Theorem 4.1 applies. This situation has been explored in [6].

We will now justify the claim that it suffices to suppose that the  $G$ -action on  $Q$  involves a single orbit type with compact stabilizer:

**Proposition 4.2.** *Suppose that, given a point  $q$  of  $Q$ , the stabilizer  $G_q$  is compact and that, given two points  $q_1$  and  $q_2$  of  $Q$ , the stabilizers  $G_{q_1}$  and  $G_{q_2}$  are conjugate in  $G$ . Then the projection  $\pi$  is a locally trivial fibration whence the orbit space  $N = Q/G$  is a smooth manifold and the projection  $\pi$  is a smooth submersion.*

*Proof.* Pick  $q \in Q$ , endow  $Q$  with a  $G_q$ -invariant Riemannian metric—such a metric arises by averaging a metric on  $Q$  over  $G_q$ —consider the induced  $G_q$ -representation on the tangent space  $T_q Q$ , and let  $V_q = (\mathfrak{g}q)^\perp$ , the orthogonal complement of the tangent space  $\mathfrak{g}q \subseteq T_q Q$  to the  $G$ -orbit of  $q$  in  $Q$ . This is the standard *infinitesimal slice* at  $q$  for the  $G$ -action on  $Q$ . Then a suitable  $G_q$ -invariant ball  $S_q \subseteq V_q$  containing the origin is a local slice, that is, the map

$$G \times_{G_q} S_q \longrightarrow Q, \quad (x, y) \mapsto x \cdot \exp_q(y), \quad x \in G, \quad y \in S_q, \quad (4.2)$$

is a local diffeomorphism onto a  $G$ -invariant neighborhood of  $q$  in  $Q$  in such a way that the subspace of  $G \times_{G_q} S_q$  which corresponds to the zero section of the associated vector bundle on  $G/G_q$  goes to the orbit  $Gq$ ; here  $\exp_q$  refers to the exponential mapping at the point  $q$  for the Riemannian metric on  $Q$ . Given a point  $p$  of  $S_q$ , the stabilizer  $G_p$  of  $p$  is plainly a subgroup of  $G_q$ ; however, in view of the hypothesis that  $G_p$  and  $G_q$  be conjugate,  $G_p$  necessarily coincides with  $G_q$ , that is, the action of  $G_q$  on  $S_q$  is trivial, whence (4.2) takes the form

$$(G/G_q) \times S_q \longrightarrow Q, \quad (x, y) \mapsto x \cdot \exp_q(y). \quad (4.3)$$

Close to the point  $q$  of  $Q$ , in a coordinate chart for  $Q$  arising from (4.3), the projection  $\pi$  amounts to the projection to  $S_q$ . This implies the assertion.  $\square$

We note that Proposition 4.2 is perfectly consistent with the classical result of *Ehresmann's* telling us that a smooth, regular and proper surjection is a locally trivial fibration. Moreover, under the circumstances of Proposition 4.2, the assignment to a vector in the Lie algebra  $\mathfrak{g}$  of its induced fundamental vector field on  $Q$  yields an exact sequence

$$\mathfrak{g} \times Q \longrightarrow TQ \longrightarrow Q \times_N TN \longrightarrow 0 \quad (4.4)$$

of smooth  $G$ -vector bundles over  $Q$ , each of the unlabelled horizontal arrows being of constant rank.

## 5 Homogeneous spaces

Let  $G$  be a Lie group,  $H$  a closed subgroup, consider the homogeneous space  $Q = G/H$  as a  $G$ -manifold as usual, and lift the  $G$ -action to  $T(G/H)$  and  $T^*(G/H)$  in the standard manner. Our aim is to determine the Poisson algebra of smooth functions on the space  $G \backslash (T^*(G/H))$  of  $G$ -orbits in  $T^*(G/H)$  or, equivalently, the Poisson algebra of smooth  $G$ -invariant functions on  $T^*(G/H)$ . We note that we do not systematically distinguish in notation between orbits relative to an action on the left and orbits relative to an action on the right.

When  $H$  is the trivial group, the Poisson algebra of smooth  $G$ -invariant functions on  $T^*(G/H)$  comes down to the ordinary Lie-Poisson algebra of smooth functions on  $\mathfrak{g}^*$ . For general  $H$ , we will suppose that  $G/H$  is reductive, cf. [21]. That is to say, as an  $H$ -module,  $\mathfrak{g}$  decomposes as a direct sum  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$  in such a way that  $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$  and that  $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}$ . From the exact sequence

$$0 \longrightarrow G \times \mathfrak{h} \longrightarrow TG \longrightarrow G \times \mathfrak{q} \longrightarrow 0 \quad (5.1)$$

of  $H$ -vector bundles over  $G$  we see that the canonical morphism

$$G \times_H \mathfrak{q} \longrightarrow T(G/H) \quad (5.2)$$

of smooth vector bundles over  $G/H$  is an isomorphism whence the algebra  $(C^\infty(T^*(G/H)))^G$  of smooth  $G$ -invariant functions on  $T^*(G/H)$  comes down to the algebra  $(C^\infty(\mathfrak{q}^*))^H$  of smooth  $H$ -invariant functions on  $\mathfrak{q}^*$ , and this algebra contains the algebra  $(\mathcal{S}[\mathfrak{q}])^H$  of  $H$ -invariant polynomials in  $\mathfrak{q}$  in an obvious manner, and  $(\mathcal{S}[\mathfrak{q}])^H$  is Fréchet dense in  $(C^\infty(\mathfrak{q}^*))^H$ . Since the decomposition of  $\mathfrak{g}$  is reductive, the induced Poisson bracket on  $(\mathcal{S}[\mathfrak{q}])^H$  is zero. Since  $(\mathcal{S}[\mathfrak{q}])^H$  is Fréchet dense in  $(C^\infty(\mathfrak{q}^*))^H$ , the Poisson structure on  $(C^\infty(T^*(G/H)))^G$  is necessarily zero as well. When  $H$  is compact, in view of a result of G. W. Schwarz [25], the algebra  $(C^\infty(\mathfrak{q}^*))^H$  of smooth  $H$ -invariant functions on  $\mathfrak{q}^*$  is the algebra of smooth functions in the generators of  $(\mathcal{S}[\mathfrak{q}])^H$ .

To reconcile this description with that given earlier we note first that the space  $N$  of  $G$ -orbits in  $Q = G/H$  comes down to a single point whence the algebra  $A$  amounts to the ground field  $\mathbb{R}$  and the extension (4.1) comes down to the identity mapping  $L \rightarrow L$  where  $L$  is the ordinary Lie algebra  $(\text{Vect}(Q))^G$  of smooth  $G$ -invariant vector fields on  $Q = G/H$ . However, inspection of the morphism

$$\begin{array}{ccc} G \times \mathfrak{q} & \longrightarrow & TQ \\ \lambda \downarrow & & \tau_Q \downarrow \\ G & \longrightarrow & Q \end{array} \quad (5.3)$$

of vector bundles shows that, as a vector space,  $(\text{Vect}(Q))^G$  amounts to the vector space  $\mathfrak{q}^H$  of  $H$ -invariants in  $\mathfrak{q}$  whence  $L \cong \mathfrak{q}^H$ . Plainly, the algebra  $(\mathcal{S}[\mathfrak{q}])^H$  of  $H$ -invariant

polynomials on  $\mathfrak{q}$  contains the algebra  $\mathcal{S}[L] \cong \mathcal{S}[\mathfrak{q}^H]$  polynomials on  $\mathfrak{q}^H$  but in general the two will *not* coincide. In particular, when the  $H$ -representation  $\mathfrak{q}$  does not contain the trivial representation,  $\mathfrak{q}^H$  is zero whereas the algebra  $(\mathcal{S}[\mathfrak{q}])^H$  of  $H$ -invariant polynomials on  $\mathfrak{q}$  is non-trivial unless  $\mathfrak{q}$  is zero since the space of  $H$ -orbits in  $\mathfrak{q}$  does not reduce to a point. Thus Theorem 3.2 cannot recover a Fréchet dense subalgebra of the algebra  $(C^\infty(T^*(G/H)))^G$  of smooth  $G$ -invariant functions on  $T^*(G/H)$ .

**Example 5.1.** Let  $G$  be a Lie group and view  $Q = G$  as a homogeneous space of the group  $G \times G$  in the standard manner, that is,  $G$  is viewed as a  $(G \times G)$ -space via left and right translation. In other words,  $H = \Delta G (\cong G)$  being the diagonal group in  $G \times G$ ,  $Q$  is identified with the space  $(G \times G)/H$  of orbits in  $G \times G$  relative to the diagonal action. The decomposition

$$\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}, \quad \mathfrak{h} = \{(X, X); X \in \mathfrak{g}\}, \quad \mathfrak{q} = \{(Y, -Y); Y \in \mathfrak{g}\}$$

is reductive, and the algebra  $(\mathcal{S}[\mathfrak{q}])^H$  of  $H$ -invariant polynomials on  $\mathfrak{q}$  comes down to the algebra  $(\mathcal{S}[\mathfrak{g}])^G$  of invariants on  $\mathfrak{g}$  (under the adjoint action). When  $\mathfrak{g}$  is semisimple,  $\mathfrak{g}^G$  is zero whereas  $(\mathcal{S}[\mathfrak{g}])^G$  is never zero unless  $\mathfrak{g}$  is zero. The Poisson bracket on  $(\mathcal{S}[\mathfrak{g}])^G$  is zero, indeed, the algebra  $(\mathcal{S}[\mathfrak{g}])^G$  is that of Casimir elements in the Lie-Poisson algebra  $\mathcal{S}[\mathfrak{g}]$ . Plainly, the method which works well in the case of a principal  $G$ -action does not suffice in the present situation, that is, Theorem 3.2 cannot recover the algebra under discussion.

**Example 5.2.** Let  $G = \mathrm{SO}(3, \mathbb{R})$  and  $H = \mathrm{SO}(2, \mathbb{R})$ , embedded into  $G$  in the standard manner, so that  $G/H$  amounts to the standard 2-sphere  $S^2$ . In the decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$  of the Lie algebra  $\mathfrak{g} = \mathfrak{so}(3, \mathbb{R})$ ,  $\dim \mathfrak{h} = 1$ , and  $\mathfrak{q}$  is the defining (irreducible) 2-dimensional representation of  $\mathrm{SO}(2, \mathbb{R})$  via rotations in the plane  $\mathfrak{q}$  the elements of which we write as  $\mathbf{x} = (x_1, x_2)$ . Since the representation  $\mathfrak{q}$  is irreducible,  $\mathfrak{q}^H$  is zero whereas the algebra  $(\mathcal{S}[\mathfrak{q}])^H$  of  $H$ -invariant polynomials on  $\mathfrak{q}$  is the polynomial algebra in the single variable  $\mathbf{x}^2 = x_1^2 + x_2^2$ . Thus, the space  $T^*S^2$  being viewed as the total space of a vector bundle on  $S^2$  with a  $\mathrm{O}(3, \mathbb{R})$ -invariant Riemannian structure, the assignment to a covector  $\alpha_q \in T^*S^2$  ( $q \in S^2$ ) of the square  $\alpha_q^2$  of its length relative to the Riemannian structure induces a diffeomorphism (beware: the notion of diffeomorphism to be suitably interpreted, see below) between  $T^*S^2/G$  and  $\mathbb{R}_{\geq 0}$ , and the Poisson bracket on the resulting algebra  $(C^\infty(T^*S^2))^G$  is zero. Here  $(C^\infty(\mathbb{R}_{\geq 0}))$  is not an algebra of ordinary smooth functions: the space  $\mathbb{R}_{\geq 0}$  is endowed with the algebra of Whitney functions relative to the embedding into  $\mathbb{R}$ ; these are continuous functions that are restrictions of smooth functions on some open interval of the kind  $] - \varepsilon, +\infty[$  where  $\varepsilon > 0$ . Again, the method which works well in the case of a principal  $G$ -action does not suffice in the present situation, that is, Theorem 3.2 cannot recover this kind of algebra under discussion.

## 6 Orthogonal representations and invariant theory

Suppose that  $Q$  is the representation space for an orthogonal representation of the Lie group  $G$ . Then, as a  $G$ -representation,  $T^*Q$  amounts to  $Q \times Q$ , endowed with the diagonal representation. The relationship between invariants upstairs and downstairs of the

cotangent bundle of  $Q$  can then be explored via invariant theory. There is no general procedure to determine the algebra of invariants of an arbitrary Lie group on a general orthogonal representation. We will therefore explain with a fundamental example how the information we are looking for can be extracted.

Let  $s \geq 1$ ,  $\ell \geq 1$ , and consider the real reductive dual pair  $(O(s, \mathbb{R}), \text{Sp}(\ell, \mathbb{R}))$  in  $\text{Sp}(s\ell, \mathbb{R})$ . Interpret the defining  $s$ -dimensional representation  $E$  of  $O(s, \mathbb{R})$  as the configuration space of a mechanical system, let  $Q = E^{\times \ell}$ , a product of  $\ell$  copies of  $E$ , endowed with the diagonal action and diagonal inner product and, as before, let  $N = Q/O(s, \mathbb{R})$  be the orbit space. This situation has been studied in the literature, cf. e. g. [18] and the references there. Let  $S^2[\mathbb{R}^\ell]$  be the real vector space of real symmetric  $(\ell \times \ell)$ -matrices and, given  $\ell$  vectors  $\mathbf{q}_1, \dots, \mathbf{q}_\ell$  in  $E$ , we will denote by  $[\mathbf{q}_j \mathbf{q}_k]$  the symmetric  $(\ell \times \ell)$ -matrix whose entries are the inner products  $\mathbf{q}_j \mathbf{q}_k$  ( $1 \leq j, k \leq \ell$ ). In view of the first main theorem of invariant theory, the association

$$Q = E^{\times \ell} \longrightarrow S^2[\mathbb{R}^\ell], (\mathbf{q}_1, \dots, \mathbf{q}_\ell) \longmapsto [\mathbf{q}_j \mathbf{q}_k] \quad (6.1)$$

is the Hilbert map of invariant theory for the  $O(s, \mathbb{R})$ -action on  $Q$ , and this map induces an injection of the orbit space  $N = Q/O(s, \mathbb{R})$  into  $S^2[\mathbb{R}^\ell]$  which identifies  $N$  with the real semi-algebraic subset of non-negative matrices in  $S^2[\mathbb{R}^\ell]$  of rank at most equal to  $s$ . Thus, when  $s$  is at least equal to  $\ell$ , the orbit space  $N = Q/O(s, \mathbb{R})$  comes down to the real semi-algebraic subset of non-negative matrices in  $S^2[\mathbb{R}^\ell]$ .

The total space  $T^*Q$  of the cotangent bundle on  $Q$ , endowed with the lifted  $O(s, \mathbb{R})$ -action amounts to the product  $(T^*E)^{\times \ell}$  of  $\ell$  copies of the total space  $T^*E$  of the cotangent bundle on  $E$ . Under the identification of  $T^*E$  with  $E \times E$  via the inner product on  $E$ , the lifted  $O(s, \mathbb{R})$ -action on  $T^*E$  comes down to the diagonal action on  $E \times E$  whence the lifted  $O(s, \mathbb{R})$ -action on  $T^*Q$  comes down to the diagonal action on the product of  $2\ell$  copies of  $E$ . This action yields an embedding of  $O(s, \mathbb{R})$  into  $\text{Sp}(s\ell, \mathbb{R})$  which realizes the dual pair  $(O(s, \mathbb{R}), \text{Sp}(\ell, \mathbb{R}))$  in  $\text{Sp}(s\ell, \mathbb{R})$ ; indeed  $O(s, \mathbb{R})$  and  $\text{Sp}(\ell, \mathbb{R})$  centralize each other in  $\text{Sp}(s\ell, \mathbb{R})$ . Let

$$\mu_{\text{Sp}(\ell, \mathbb{R})}: T^*Q \longrightarrow \mathfrak{sp}(\ell, \mathbb{R})^* \quad (6.2)$$

be the  $\text{Sp}(\ell, \mathbb{R})$ -momentum mapping. It is well known—this is a consequence of the first main theorem of invariant theory—that  $\mu_{\text{Sp}(\ell, \mathbb{R})}$  amounts to the Hilbert map of invariant theory for the  $O(s, \mathbb{R})$ -action on  $T^*Q$  and that, once  $\mathfrak{sp}(\ell, \mathbb{R})$  has been identified with its dual by means of an appropriate invariant symmetric bilinear form,  $\mu_{\text{Sp}(\ell, \mathbb{R})}$  induces an injection of the orbit space  $(T^*Q)/O(s, \mathbb{R})$  onto the real semi-algebraic subset of  $\mathfrak{sp}(\ell, \mathbb{R})$  which consists of non-negative matrices in  $\mathfrak{sp}(\ell, \mathbb{R})$  of rank at most equal to  $\min(s, \ell)$ . For intelligibility we recall that, after the identification of  $\mathfrak{sp}(\ell, \mathbb{R})$  with its dual,  $\mu_{\text{Sp}(\ell, \mathbb{R})}$  is given by the association

$$T^*Q \cong E^{\times 2\ell} \longrightarrow \mathfrak{sp}(\ell, \mathbb{R}), (\mathbf{q}_1, \mathbf{p}_1, \dots, \mathbf{q}_\ell, \mathbf{p}_\ell) \longmapsto \begin{bmatrix} \mathbf{q}_j \mathbf{p}_k & -\mathbf{q}_j \mathbf{q}_k \\ \mathbf{p}_j \mathbf{p}_k & -\mathbf{q}_j \mathbf{p}_k \end{bmatrix}. \quad (6.3)$$

See e. g. Theorem 5.4.1 in [15] and its proof for details (but the result is classical). In view of the result of G. W. Schwarz [25] quoted earlier, the algebra of continuous functions on the orbit space  $(T^*Q)/O(s, \mathbb{R})$  given by  $(C^\infty(T^*Q))^{O(s, \mathbb{R})}$  gets identified with that of smooth functions in the variables  $\mathbf{q}_j \mathbf{p}_k$ ,  $\mathbf{q}_j \mathbf{q}_k$ ,  $\mathbf{p}_j \mathbf{p}_k$  ( $1 \leq j, k \leq \ell$ ), and these

are the Whitney smooth functions relative to the embedding into  $\mathfrak{sp}(\ell, \mathbb{R})^*$ . Moreover, the momentum mapping property tells us that the Poisson structure on this algebra is induced from the Lie bracket on  $\mathfrak{sp}(\ell, \mathbb{R})$ . In more down to earth terms, cf. e. g. Section 5 of [15]: The ordinary symplectic Poisson brackets  $\{q_{j,\alpha}, p_{k,\beta}\} = \delta_{\alpha,\beta} \delta_{j,k}$  etc. on  $E^{\times 2\ell}$ , evaluated in the quadratic polynomials  $\mathbf{q}_j \mathbf{p}_k$ ,  $\mathbf{q}_j \mathbf{q}_k$ ,  $\mathbf{p}_j \mathbf{p}_k$  ( $1 \leq j, k \leq \ell$ ,  $1 \leq \alpha, \beta \leq s$ ), simply recover the Lie algebra  $\mathfrak{sp}(\ell, \mathbb{R})$ . It is worthwhile noting at this stage that the sum  $T = \mathbf{p}_1^2 + \cdots + \mathbf{p}_\ell^2$  can be interpreted as an  $O(s, \mathbb{R})$ -invariant kinetic energy term of the kind mentioned in the introduction.

In particular, when  $s$  is at least equal to  $\ell$ , the orbit space  $(T^*Q)/O(s, \mathbb{R})$  comes down to the real semi-algebraic subset of  $\mathfrak{sp}(\ell, \mathbb{R})$  which consists of non-negative matrices in  $\mathfrak{sp}(\ell, \mathbb{R})$  of rank at most equal to  $\ell$ .

To compare this description with the statement of Theorem 4.1—we note that the hypothesis that the projection from  $Q$  to  $N$  be regular is not satisfied at this point but this is of no account for the present discussion—, let  $A$  be the algebra of continuous functions on the orbit space  $N = Q/O(s, \mathbb{R})$  given by  $(C^\infty(Q))^{O(s, \mathbb{R})}$ ; just as before, in view of the quoted result of G. W. Schwarz [25], this algebra amounts to the algebra of smooth functions in the variables  $\mathbf{q}_j \mathbf{q}_k$  ( $1 \leq j, k \leq \ell$ ) and coincides with that of Whitney smooth functions relative to the embedding of  $N$  into  $S^2[\mathbb{R}^\ell]$ . As an  $A$ -module, the  $(\mathbb{R}, A)$ -Lie algebra  $L$  is generated by the  $\mathbf{q}_j \mathbf{p}_k$ 's ( $1 \leq j, k \leq \ell$ ). Thus the Poisson algebra  $\mathcal{S}_A[L]$  is the Poisson subalgebra of  $(C^\infty(T^*Q))^{O(s, \mathbb{R})}$  generated by  $A$  and  $\mathbf{q}_j \mathbf{p}_k$  ( $1 \leq j, k \leq \ell$ ) and, since  $\mathcal{S}_A[L]$  does not contain the generators  $\mathbf{p}_j \mathbf{p}_k$  ( $1 \leq j, k \leq \ell$ ), the Poisson algebra  $\mathcal{S}_A[L]$  *cannot recover* the entire Poisson algebra  $(C^\infty(T^*Q))^{O(s, \mathbb{R})}$ , that is,  $\mathcal{S}_A[L]$  cannot be dense in  $(C^\infty(T^*Q))^{O(s, \mathbb{R})}$ , since the coordinate functions  $\mathbf{p}_j \mathbf{p}_k$  are missing. On the other hand, we have noted above that, suitably interpreted, these terms include the kinetic energy, so they cannot be ignored.

We will now exploit this discussion to illustrate Theorem 4.1: Let  $G = O(3, \mathbb{R})$  act on  $\mathbb{R}^3$  by the standard action and let  $Q = \mathbb{R}^3 \setminus \{0\}$ . The hypotheses of Theorem 4.1 are satisfied, and  $T^*Q$  is diffeomorphic to  $Q \times \mathbb{R}^3$ , endowed with the diagonal action. Furthermore, the orbit space  $Q/G$  comes down to  $\mathbb{R}_{>0}$ , that is,  $A$  is the algebra of smooth functions on the interval  $]0, +\infty[$ ; these are the smooth functions in the variable  $\mathbf{q}^2$ , ( $\mathbf{q} \in Q$ ). Let  $S^2[\mathbb{R}^2]$  be the space of real symmetric  $(2 \times 2)$ -matrices. The invariant theory used above tells us that the association

$$\mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow S^2[\mathbb{R}^2], (\mathbf{q}, \mathbf{p}) \longmapsto \begin{bmatrix} \mathbf{q}^2 & \mathbf{qp} \\ \mathbf{qp} & \mathbf{p}^2 \end{bmatrix} \quad (6.4)$$

identifies the orbit space  $(\mathbb{R}^3 \times \mathbb{R}^3)/G$  with the space  $S^2[\mathbb{R}^2]_{\geq 0}$  of non-negative real symmetric  $(2 \times 2)$ -matrices and hence the orbit space  $(Q \times \mathbb{R}^3)/G$  with the corresponding subspace of  $S^2[\mathbb{R}^2]_{\geq 0}$ . It is then straightforward to write out the brackets between the three generators

$$\mathbf{q}^2, \mathbf{qp}, \mathbf{p}^2, \quad (6.5)$$

of the various algebras of functions on the various orbit spaces. Indeed, as noted above for the more general situation where  $\ell$  is arbitrary rather than just  $\ell = 1$ , cf. also e. g. Section 5 of [15], the (real span of the) brackets between the three generators (6.5) come(s) down to the Lie algebra  $\mathfrak{sp}(1, \mathbb{R}) (= \mathfrak{sl}(2, \mathbb{R}))$  whence, cf. Theorem 5.4.1 in [15], the real



algebraic quotient  $(\mathbb{R}^3 \times \mathbb{R}^3)/G$  (that is, the real affine variety having the algebra of  $G$ -invariants  $(\mathbb{R}[\mathbb{R}^3 \times \mathbb{R}^3])^G$  as its real coordinate ring) comes down to a copy of  $\mathbb{R}^3$ , spanned by (6.5), with Poisson algebra  $\mathcal{S}[\mathfrak{sp}(1, \mathbb{R})]$ . As a real semi-algebraic quotient, the orbit space  $(\mathbb{R}^3 \times \mathbb{R}^3)/G$  is then the image of  $\mathbb{R}^3 \times \mathbb{R}^3$  under the projection to  $(\mathbb{R}^3 \times \mathbb{R}^3)/G \cong \mathbb{R}^3$  which, under the present circumstances, comes down to the matrices in  $S^2[\mathbb{R}^2]$  that are non-negative. As an  $A$ -module, the  $(\mathbb{R}, A)$ -Lie algebra  $L$  is generated by  $\mathbf{qp}$ . Thus the Poisson algebra  $\mathcal{S}_A[L]$  is the Poisson subalgebra of  $(C^\infty(T^*Q))^G$  generated by  $A$  and  $\mathbf{qp}$  and, since  $\mathcal{S}_A[L]$  does not contain the generator  $\mathbf{p}^2$ , the Poisson algebra  $\mathcal{S}_A[L]$  does *not recover* the Poisson algebra  $(C^\infty(T^*Q))^G$ , i. e. does not map onto an algebra that is dense in  $(C^\infty(T^*Q))^G$  in the appropriate Fréchet topology. Thus the method which works well in the case of a principal  $G$ -action does not suffice in the present situation, that is, Theorem 4.1 cannot yield a complete description of the Poisson structure on the orbit space  $(T^*Q)/G$ . In particular, this method would ignore the kinetic energy term  $\mathbf{p}^2$ .

## 7 Reduction to invariant theory

Let  $Q$  be a smooth manifold, let  $G$  be a compact Lie group, and suppose that  $G$  acts smoothly on  $Q$  with finitely many orbit types. The equivariant version of Whitney's embedding theorem, the Mostov-Palais theorem, cf. [20], [22], tells us that the manifold  $Q$  admits a  $G$ -equivariant embedding into some orthogonal representation space  $E$  for  $G$ .

The embedding of  $Q$  into  $E$  lifts to a  $G$ -equivariant embedding of  $T^*Q$  onto a symplectic  $G$ -submanifold of  $T^*E$ , that is, onto a  $G$ -invariant second class constraint submanifold. The Poisson algebra on the quotient  $(T^*E)/G$  can be understood in terms of invariant theory, special cases of which have been explored in Section 6 above. The Poisson bracket on  $T^*Q$  can then be described in terms of a  $G$ -invariant *Dirac*-bracket on  $T^*E$ , and this bracket descends to a Dirac bracket on  $(T^*E)/G$ . Equivariant equations for the embedding of  $Q$  into  $E$  and their derivatives yield equivariant equations for the embedding of  $T^*Q$  into  $T^*E$  and these, combined with the Dirac brackets on  $(T^*E)/G$ , will eventually yield the Poisson bracket on the orbit space  $(T^*Q)/G$ . We shall explain the details at another occasion.

We conclude with a brief discussion of a special case: Let  $K$  be a compact connected Lie group, viewed as a  $K$ -space via the conjugation action of  $K$  on itself, lift this action to the total space  $T^*K$  of the cotangent bundle of  $K$ , let  $\mathfrak{k}$  denote the Lie algebra of  $K$ , and let  $\mu: T^*K \rightarrow \mathfrak{k}^*$  be the corresponding  $K$ -equivariant momentum mapping, normalized so as to have the value zero at the zero covector at the neutral element  $e$  of  $K$ . In [17], we have applied the kind of procedure sketched in the present section to the zero momentum reduced space  $(T^*K)_0 = \mu^{-1}(0)/K$ . In this particular case,  $T^*K$  is canonically diffeomorphic to the complexified group  $K^\mathbb{C}$  via the polar map and a choice of invariant inner product on  $\mathfrak{k}$  of  $K$ , and the zero momentum reduced space is homeomorphic to the affine algebraic quotient  $K^\mathbb{C}/K^\mathbb{C}$  and thus acquires a stratified Kähler structure [17]. In particular, in this case, the real and imaginary parts of the characters of the fundamental representations yield functions in  $C^\infty((T^*K)_0)$  (beware: this is not an algebra of ordinary smooth functions), and the main result of [17] says that, for  $K = \mathrm{U}(n), \mathrm{SU}(n), \mathrm{Sp}(n), \mathrm{SO}(2n+1, \mathbb{R}), G_{2(-14)}$ , as a Poisson algebra, the real coordi-

nate ring of  $(T^*K)_0$  is generated by the real and imaginary parts of the characters of the fundamental representations and their iterated Poisson brackets. By construction, the real and imaginary parts of the characters of the fundamental representations yield (continuous) functions on the orbit space  $(T^*K)/K$ . Furthermore, the algebra  $(\mathcal{S}[\mathfrak{k}])^K$  of  $K$ -invariant polynomials in  $\mathfrak{k}$  boils down to the algebra of  $K$ -invariant polynomial functions on  $\mathfrak{k}^*$  and hence induces, via the momentum mapping  $\mu$ , a class of functions on the orbit space  $(T^*K)/K$  which belong to the algebra  $C^\infty((T^*K)/K)$  of continuous functions on  $(T^*K)/K$  given by the algebra  $(C^\infty(T^*K))^K$  of ordinary smooth  $K$ -invariant functions on  $T^*K$ ; relative to the induced Poisson bracket on  $C^\infty((T^*K)/K)$ , these functions are plainly Casimir functions. Thus these functions, together with the real and imaginary parts of the characters of the fundamental representations, yield a huge class of functions on  $(T^*K)/K$  and, in view of the quoted result in [17], it is reasonable to conjecture that, for  $K = U(n), SU(n), Sp(n), SO(2n+1, \mathbb{R}), G_{2(-14)}$ , as a Poisson algebra, a suitably defined real coordinate ring of  $(T^*K)/K$  is generated by these functions. It would then be interesting to determine the Poisson structure on the quotient  $(T^*K)/K$  explicitly. We hope to return to these issues elsewhere.

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